Center-Outward Sign- and Rank-Based Quadrant, Spearman, and Kendall Tests for Multivariate Independence

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Abstract

Defining multivariate generalizations of the classical univariate ranks has been a long-standing open problem in statistics. Optimal transport has been shown to offer a solution by transporting data points to grid points approximating a reference measure (Chernozhukov et al. 2017; Hallin 2017; Hallin et al. 2021). We take up this new perspective to develop and study multivariate analogues of popular correlations measures including the sign covariance, Kendall's tau, and Spearman's rho. Our tests are genuinely distribution-free, hence valid irrespective of the actual (absolutely continuous) distributions of the observations. We present asymptotic distribution theory for these new statistics, providing asymptotic approximations to critical values to be used for testing independence as well as a power analysis of the resulting tests. Interestingly, we are able to establish a multivariate elliptical Chernoff–Savage property, which guarantees that, under ellipticity, our nonparametric tests of independence when compared to Gaussian procedures enjoy an asymptotic relative efficiency of one or larger. Hence, the nonparametric tests constitute a safe replacement for procedures based on multivariate Gaussianity.

1 Introduction

The problem of testing for independence between two random variables with unspecified densities has been among the very first applications of rank-based methods in statistical inference. Spearman's correlation coefficient was proposed in the early 1900s (Spearman 1904), and Kendall's rank correlation goes back to Kendall (1938), long before Wilcoxon (1945) gave his rank sum and signed rank tests for location.

The multivariate version of the same problem—testing independence between two random vectors with unspecified densities—is significantly harder, crucially due to the difficulty of defining a multivariate counterpart to univariate ranks. Indeed, for $d > 1$ the real space $\mathbb{R}^d$ lacks a canonical ordering. As a result, the problem of defining, in dimension $d > 1$, concepts of signs and ranks enjoying the properties that make the traditional ranks so successful in univariate statistical inference has been an open problem for more than half a century. One of the most important properties is the exact distribution-freeness (for i.i.d. samples from absolutely continuous distributions). In an important new development involving optimal transport, the concept of center-outward ranks and signs was proposed recently by Chernozhukov et al. (2017), Hallin (2017), and Hallin et al. (2021) and enjoys a property of "maximal distribution-freeness", contrary to earlier concepts put forth in work such as Puri and Sen (1971); Oja (2010); Liu and Singh (1993); Zuo and He (2006); Hallin and Paindaveine (2002b,a).

For testing independence between two random vectors, the first attempt to provide a rank-based alternative to the Gaussian likelihood ratio method of Wilks (1935) was developed in Chapter 8 of Puri and Sen (1971) and, for almost thirty years, has remained the only rank-based approach to the problem. The proposed tests, however, are based on componentwise rankings and are not distribution-free—unless, of course, both vectors have dimension one, in which case we are back to the traditional context of bivariate independence (see, e.g., Chapter III.6 of Hájek and Šidák (1967)). This issue persists in more recent work, e.g., that of Puri and Sen (1971), Randles (1989), Gieser (1993), Gieser and Randles (1997), Taskinen, Kankainen, and Oja (2003, 2004), and Taskinen, Oja, and Randles (2005).

We note here that the above work does provide test statistics that are asymptotically distribution-free in subclasses such as elliptical distributions. From the perspective we take here, such subclasses are too restrictive. Moreover, there is a crucial difference between finite-sample and asymptotic distribution-freeness. Indeed, one should be wary that a sequence of tests $\psi^{(n)}$ with asymptotic size $\lim_{n\to\infty} \mathrm{E}_{\mathrm{P}}[\psi^{(n)}] = \alpha$ under any element P in a class $\mathcal{P}$ of distributions does not necessarily have asymptotic size α under unspecified $\mathrm{P} \in \mathcal{P}$: the convergence of $\mathrm{E}_{\mathrm{P}}[\psi^{(n)}]$ to α, indeed, typically is not uniform over $\mathcal{P}$, so that, in general, $\lim_{n\to\infty} \sup_{\mathrm{P}\in\mathcal{P}} \mathrm{E}_{\mathrm{P}}[\psi^{(n)}] \neq \alpha$. Genuinely distribution-free tests $\psi^{(n)}$, where $\mathrm{E}_{\mathrm{P}}[\psi^{(n)}]$ does not depend on P, do not suffer that problem, and this is why finite-sample distribution-freeness is a fundamental property.

Palliating these limitations of the existing procedures by defining genuinely distribution-free—now over the class of all absolutely continuous distributions—multivariate extensions of the quadrant, Spearman, and Kendall tests, based on the concept of center-outward ranks and signs, is thus highly desirable. It is the objective of this paper.

While this paper is focusing on quadrant, Spearman, and Kendall tests of independence, other tests have been considered in the literature. Center-outward ranks and signs have been used recently by Shi, Drton, and Han (2021+) in the construction of distribution-free versions of distance covariance tests for multivariate independence, and a general framework for designing distribution-free tests of multivariate independence that are consistent and statistically efficient based on center-outward ranks and signs has been developed in Shi et al. (2021+). Multivariate ranks (based on measure transportation to the unit cube rather than the unit ball) have been used similarly in Ghosal and Sen (2021+), Deb and Sen (2021+).

Center-outward ranks and signs also have been used successfully in other statistical problems: rank tests and R-estimation for VARMA models (Hallin, La Vecchia, and Liu 2021+, 2022+), rank tests for multiple-output regression and MANOVA (Hallin, Hlubinka, and Hudecová 2022+), and two-sample goodness-of-fit tests (Deb and Sen 2021+; Deb, Bhattacharya, and Sen 2021; Hallin and Mordant 2021). We show here how center-outward ranks and signs naturally allow us to define distribution-free multivariate versions of the popular quadrant, Spearman, and Kendall tests.

The paper is organized as follows. Section 2 briefly reviews the notion of center-outward ranks and signs, and Section 3 introduces our tests of multivariate independence based on center-outward ranks and signs. In Section 4, we establish an elliptical Chernoff–Savage property for our center-outward test based on van der Waerden scores, which uniformly dominates, against Konijn alternatives, Wilks' test for multivariate independence, and we also derive an analog of Hodges and Lehmann (1956)'s result for the problem under study. This paper ends with a short conclusion in Section 5. All the detailed proofs of our results are available upon request from the authors.

2 Center-outward distribution functions, ranks, and signs

2.1 Definitions

Denoting by $\mathbb{S}_d$ and $\mathcal{S}_{d-1}$, respectively, the open unit ball and the unit hypersphere in $\mathbb{R}^d$, let U_d stand for the spherical[1] uniform distribution over $\mathbb{S}_d$. Let P belong to the class $\mathcal{P}_d$ of Lebesgue-absolutely continuous distributions over $\mathbb{R}^d$. The main result in McCann (1995) then implies the existence of an a.e. unique convex (and lower semi-continuous) function $\phi : \mathbb{R}^d \to \mathbb{R}$ with gradient $\nabla\phi$ such that[2] $\nabla\phi\#\mathrm{P} = \mathrm{U}_d$. Call *center-outward distribution function* of P any version $\mathbf{F}_\pm$ of this a.e. unique gradient.

Further properties of $\mathbf{F}_\pm$ require further regularity assumptions. Assume that P is in the so-called class $\mathcal{P}_d^+ \subset \mathcal{P}_d$ of distributions *with nonvanishing densities*—namely, the class of distributions with density $f := \mathrm{dP}/\mathrm{d}\mu_d$ (μ_d the d-dimensional Lebesgue measure) such that, for all $D \in \mathbb{R}^+$, there exist constants $\lambda^-_{D;\mathrm{P}}$ and $\lambda^+_{D;\mathrm{P}}$ satisfying

$$0 < \lambda^-_{D;\mathrm{P}} \leq f(\mathbf{z}) \leq \lambda^+_{D;\mathrm{P}} < \infty \tag{2.1}$$

for all $\mathbf{z}$ with $\|\mathbf{z}\| \leq D$.

Then, it follows from Figalli (2018) that there exists a version of $\mathbf{F}_\pm$ defining a homeomorphism between the punctured unit ball $\mathbb{S}_d \setminus \{\mathbf{0}\}$ and $\mathbb{R}^d \setminus \mathbf{F}_\pm^{-1}(\{\mathbf{0}\})$; that version has a continuous inverse $\mathbf{Q}_\pm$ (with domain $\mathbb{S}_d \setminus \{\mathbf{0}\}$), which naturally qualifies as P's *center-outward quantile function*. Figalli's result is extended, in del Barrio, González-Sanz, and Hallin (2020), to a more general[3] class $\mathcal{P}_d^\pm$ of absolutely continuous distributions, while the definition of $\mathbf{F}_\pm$ given in Hallin et al. (2021) aims at selecting, for each $\mathrm{P} \in \mathcal{P}_d$, a version of $\nabla\phi$ which, whenever $\mathrm{P} \in \mathcal{P}_d^\pm$, is yielding that homeomorphism. For the sake of simplicity, since we are not interested in quantiles, we stick here to the a.e. unique definition given above for $\mathrm{P} \in \mathcal{P}_d$, and, whenever asymptotic statements are made, to $\mathrm{P} \in \mathcal{P}_d^+$.

Turning to sample quantities, denote by $\mathbf{Z}^{(n)}$ a triangular array $\big(\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_n^{(n)}\big)$, $n \in \mathbb{N}$ of i.i.d. d-dimensional random vectors with distribution P. Associated with $\mathbf{Z}^{(n)}$ is the *empirical center-outward distribution function* $\mathbf{F}_\pm^{(n)}$ mapping the n-tuple $\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_n^{(n)}$ to a "regular" grid $\mathfrak{G}_n$ of the unit ball $\mathbb{S}_d$. That regular grid $\mathfrak{G}_n$ is obtained as follows:

(a) first factorize n into $n = n_R n_S + n_0$, with $0 \leq n_0 < \min(n_R, n_S)$;[4]

(b) next consider a "regular array" $\mathfrak{S}_{n_S} := \{\mathbf{s}_1^{n_S}, \ldots, \mathbf{s}_{n_S}^{n_S}\}$ of n_S points on the sphere $\mathcal{S}_{d-1}$ (see the comment below);

(c) construct the grid consisting in the collection $\mathfrak{G}_n$ of the $n_R n_S$ points $\mathfrak{g}$ of the form

$$\big(r/\big(n_R+1\big)\big)\mathbf{s}_s^{n_S}, \quad r = 1, \ldots, n_R,\ s = 1, \ldots, n_S,$$

along with (n_0 copies of) the origin in case $n_0 \neq 0$: in total $n - (n_0 - 1)$ or n distinct points, thus, according as $n_0 > 0$ or $n_0 = 0$.

By "regular" we mean "as regular as possible", in the sense, for example of the *low-discrepancy sequences* of the type considered in numerical integration, Monte-Carlo methods, and experimental design.[5] The only mathematical requirement needed for the asymptotic results below is the weak convergence, as $n_S \to \infty$, of the uniform discrete distribution over $\mathfrak{S}_{n_S}$ to the uniform distribution over $\mathcal{S}_{d-1}$. A uniform i.i.d. sample of points over $\mathcal{S}_{d-1}$ (almost surely) satisfies such a requirement. However, one easily can construct arrays that are "more regular" than an i.i.d. one. For instance, one could see that n_S or $n_S - 1$ of the points in $\mathfrak{G}_n$ are such that $-\mathbf{s}_s^{n_S}$ also belongs to $\mathfrak{S}_{n_S}$, so that $\|\sum_{s=1}^{n_S} \mathbf{s}_s^{n_S}\| = 0$ or 1 according as n_S is even or odd. One also could consider

[1]Namely, the spherical distribution with uniform (over $[0, 1]$) radial density—equivalently, the product of a uniform over the distances to the origin and a uniform over the unit sphere $\mathcal{S}_{d-1}$. For $d = 1$, U_1 coincides with the Lebesgue uniform over $(-1, 1)$.

[2]We borrow from measure transportation the convenient notation $T\#\mathrm{P}$ ($T : \mathbb{R}^d \to \mathbb{R}^d$ *pushes* P *forward to* $T\#\mathrm{P}$) for the distribution of $T(\mathbf{Z})$ under $\mathbf{Z} \sim \mathrm{P}$.

[3]Namely, $\mathcal{P}_d^+ \subsetneq \mathcal{P}_d^\pm \subsetneq \mathcal{P}_d$

[4]Note that this implies that $n_0/n = o(1)$ as $n \to \infty$. See Mordant (2021, Chapter 7.4) for a suggestion of selecting n_R and n_S.

[5]See also Hallin and Mordant (2021) for a spherical version of the so-called Halton sequences.

factorizations of the form $n = n_R n_S + n_0$ with n_S even, then require $\mathfrak{G}_{n_S}$ to be symmetric with respect to the origin, yielding $\sum_{s=1}^{n_S} \mathbf{s}_s^{n_S} = \mathbf{0}$.

The empirical counterpart $\mathbf{F}_\pm^{(n)}$ of $\mathbf{F}_\pm$ is defined as the (bijective, once the origin is given multiplicity n_0) mapping from $\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_n^{(n)}$ to the grid $\mathfrak{G}_n$ that minimizes $\sum_{i=1}^n \left\|\mathbf{F}_\pm^{(n)}(\mathbf{Z}_i^{(n)}) - \mathbf{Z}_i^{(n)}\right\|^2$. That mapping is unique with probability one; in practice, it is obtained via a simple optimal assignment (pairing) algorithm (a linear program; see Hallin et al. (2021) for details). Call *center-outward rank* of $\mathbf{Z}_i^{(n)}$ the integer (in $\{1, \ldots, n_R\}$ or $\{0, \ldots, n_R\}$ according as $n_0 = 0$ or not)

$$R_{i;\pm}^{(n)} := (n_R + 1)\left\|\mathbf{F}_\pm^{(n)}(\mathbf{Z}_i^{(n)})\right\| \quad i = 1, \ldots, n$$

and *center-outward sign* of $\mathbf{Z}_i^{(n)}$ the unit vector

$$\mathbf{S}_{i;\pm}^{(n)} := \mathbf{F}_\pm^{(n)}(\mathbf{Z}_i^{(n)}) / \left\|\mathbf{F}_\pm^{(n)}(\mathbf{Z}_i^{(n)})\right\| \quad \text{for } \mathbf{F}_\pm^{(n)}(\mathbf{Z}_i^{(n)}) \neq \mathbf{0};$$

put $\mathbf{S}_{i;\pm}^{(n)} = \mathbf{0}$ for $\mathbf{F}_\pm^{(n)}(\mathbf{Z}_i^{(n)}) = \mathbf{0}$.

Some desirable finite-sample properties, such as strict independence between the ranks and the signs, only hold for $n_0 = 0$ or 1, due to the fact that the mapping from the sample to the grid is no longer injective for $n_0 \geq 2$. This, which has no asymptotic consequences (since the number n_0 of tied values involved is $o(n)$ as $n \to \infty$), is easily taken care of by the following tie-breaking device:

(i) randomly select n_0 directions $\mathbf{s}_1^0, \ldots, \mathbf{s}_{n_0}^0$ in $\mathfrak{G}_{n_S}$, then

(ii) replace the n_0 copies of the origin with the new gridpoints

$$[1/2(n_R + 1)]\mathbf{s}_1^0, \ldots, [1/2(n_R + 1)]\mathbf{s}_{n_0}^0. \qquad (2.2)$$

The resulting grid (for simplicity, the same notation $\mathfrak{G}_n$ is used) no longer has multiple points, and the optimal pairing between the sample and this grid is bijective; the n_0 smallest ranks, however, take the non-integer value $1/2$.

2.2 Main properties

This section summarizes some of the main properties of the concepts defined in Sections 2.1; further properties and the proofs can be found in Hallin et al. (2021), Hallin, Hlubinka, and Hudecová (2022+) and Hallin (2022).

Proposition 2.1. *Let $\mathbf{F}_\pm$ denote the center-outward distribution function of $\mathrm{P} \in \mathcal{P}_d$. Then,*

(i) $\mathbf{F}_\pm$ *is a probability integral transformation of $\mathbb{R}^d$: namely, $\mathbf{Z} \sim \mathrm{P}$ iff $\mathbf{F}_\pm(\mathbf{Z}) \sim \mathrm{U}_d$; by construction, $\|\mathbf{F}_\pm(\mathbf{Z})\|$ is uniform over $[0, 1)$, $\mathbf{F}_\pm(\mathbf{Z})/\|\mathbf{F}_\pm(\mathbf{Z})\|$ is uniform over the sphere $\mathcal{S}_{d-1}$, and they are mutually independent.*

Let $\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_n^{(n)}$ be i.i.d. with distribution $\mathrm{P} \in \mathcal{P}_d$ and center-outward distribution function $\mathbf{F}_\pm$. Then,

(ii) $\left(\mathbf{F}_\pm^{(n)}(\mathbf{Z}_1^{(n)}), \ldots, \mathbf{F}_\pm^{(n)}(\mathbf{Z}_n^{(n)})\right)$ *is uniformly distributed over the $n!/n_0!$ permutations with repetitions of the gridpoints in $\mathfrak{G}_n$ with the origin counted as n_0 indistinguishable points (resp. the $n!$ permutations of $\mathfrak{G}_n$ if either $n_0 \leq 1$ or the tie-breaking device described in Section 2.1 is adopted);*

(iii) if either $n_0 = 0$ or the tie-breaking device described in Section 2.1 is adopted, the n-tuple of center-outward ranks $\left(R_{1;\pm}^{(n)}, \ldots, R_{n;\pm}^{(n)}\right)$ and the n-tuple of center-outward signs $\left(\mathbf{S}_{1;\pm}^{(n)}, \ldots, \mathbf{S}_{n;\pm}^{(n)}\right)$ are mutually independent;

(iv) if either $n_0 \leq 1$ or the tie-breaking device described in Section 2.1 is adopted, $\left(\mathbf{F}_\pm^{(n)}(\mathbf{Z}_1^{(n)}), \ldots, \mathbf{F}_\pm^{(n)}(\mathbf{Z}_n^{(n)})\right)$ is strongly essentially maximal ancillary.[6]

Assuming, moreover, that $\mathrm{P} \in \mathcal{P}_d^+$,

(v) (Glivenko–Cantelli)

$$\max_{1 \leq i \leq n} \left\|\mathbf{F}_\pm^{(n)}(\mathbf{Z}_i^{(n)}) - \mathbf{F}_\pm(\mathbf{Z}_i^{(n)})\right\| \to 0 \text{ a.s.} \quad \textit{as } n \to \infty.$$

Center-outward distribution functions, ranks, and signs also inherit, from the invariance of squared Euclidean distances, elementary but quite remarkable invariance and equivariance properties under orthogonal transformations and global rescaling. Denote by $\mathbf{F}_\pm^{\mathbf{Z}}$ the center-outward distribution function of $\mathbf{Z}$ and by $\mathbf{F}_\pm^{\mathbf{Z};(n)}$ the empirical distribution function of an i.i.d. sample $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ associated with a grid $\mathfrak{G}_n$.

Proposition 2.2. *Let $\boldsymbol{\mu} \in \mathbb{R}^d$, $k \in \mathbb{R}^+$, and denote by $\mathbf{O}$ a $d \times d$ orthogonal matrix. Then,*

(i) $\mathbf{F}_\pm^{\boldsymbol{\mu}+k\mathbf{OZ}}(\boldsymbol{\mu} + \mathbf{Oz}) = \mathbf{OF}_\pm^{\mathbf{Z}}(\mathbf{z})$, $\mathbf{z} \in \mathbb{R}^d$;

(ii) denoting by $\mathbf{F}_\pm^{\boldsymbol{\mu}+k\mathbf{OZ};(n)}$ the empirical distribution function of the sample $\boldsymbol{\mu} + k\mathbf{OZ}_1, \ldots, \boldsymbol{\mu} + k\mathbf{OZ}_n$ associated with the grid $\mathbf{O}\mathfrak{G}_n$ (hence by $\mathbf{F}_\pm^{\mathbf{Z};(n)}$ the empirical distribution function of the sample $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ associated with the grid $\mathfrak{G}_n$),

$$\mathbf{F}_\pm^{\boldsymbol{\mu}+k\mathbf{OZ};(n)}(\boldsymbol{\mu}+k\mathbf{OZ}_i) = \mathbf{OF}_\pm^{\mathbf{Z};(n)}(\mathbf{Z}_i), \ \ i = 1, \ldots, n. \qquad (2.3)$$

Note that the orthogonal transformations in Proposition 2.2 include the permutations of $\mathbf{Z}$'s components. Invariance with respect to such permutations is an essential requirement for hypothesis testing in multivariate analysis.

3 Rank-based tests for multivariate independence

3.1 Center-outward test statistics for multivariate independence

In this section, we describe the test statistics we are proposing for testing independence between two random vectors. Consider a sample

$$(\mathbf{X}_{11}', \mathbf{X}_{21}')', (\mathbf{X}_{12}', \mathbf{X}_{22}')', \ldots, (\mathbf{X}_{1n}', \mathbf{X}_{2n}')'$$

of n i.i.d. copies of some $(d_1 + d_2) = d$-dimensional random vector $(\mathbf{X}_1', \mathbf{X}_2')'$ with Lebesgue-absolutely continuous distribution $\mathrm{P} \in \mathcal{P}_d$ and density f. We are interested in the null hypothesis under which $\mathbf{X}_1$ and $\mathbf{X}_2$, with unspecified marginal distributions P_1 (density f_1) and P_2 (density f_2), respectively, are mutually independent: f then factorizes into $f = f_1 f_2$.

[6] See Section 2.4 and Appendices D.1 and D.2 of Hallin et al. (2021) for a precise definition and a proof of this essential property.

Denote by $R^{(n)}_{ki;\pm}$ and $\mathbf{S}^{(n)}_{ki;\pm}$, $i = 1, 2, \ldots, n$ the center-outward rank and the sign of $\mathbf{X}_{ki}$ computed from $\mathbf{X}_{k1}, \mathbf{X}_{k2}, \ldots, \mathbf{X}_{kn}$, $k = 1, 2$, respectively. For the simplicity of notation, assume, without loss of generality as $n \to \infty$, that the grid used for computing those ranks and signs is such that $\sum_{s=1}^{n_S} \mathbf{s}_s^{n_S} = \mathbf{0}$, for $d = d_1, d_2$. Also assume that $n_0 = 0$ or 1 (if necessary, after implementing the tie-breaking device described in Section 2.1). This implies that $\sum_{i=1}^{n} \mathbf{S}^{(n)}_{ki;\pm} = \mathbf{0}$ for $k = 1, 2$, and moreover, that

$$\sum_{i=1}^{n} J_k\big(R^{(n)}_{ki;\pm}/(n_R + 1)\big)\mathbf{S}^{(n)}_{ki;\pm} = \mathbf{0}$$

for any *score functions* $J_k : [0, 1) \to \mathbb{R}$, $k = 1, 2$.

Consider the $d_1 \times d_2$ matrices

$$\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{S}^{(n)}_{1i;\pm}\mathbf{S}^{(n)\prime}_{2i;\pm}, \tag{3.1}$$

$$\underset{\sim}{\mathbf{W}}^{(n)}_{\text{S}} := \frac{1}{n(n_R+1)^2}\sum_{i=1}^{n} R^{(n)}_{1i;\pm}R^{(n)}_{2i;\pm}\mathbf{S}^{(n)}_{1i;\pm}\mathbf{S}^{(n)\prime}_{2i;\pm}, \tag{3.2}$$

$$\underset{\sim}{\mathbf{W}}^{(n)}_{\text{K}} := \binom{n}{2}^{-1}\sum_{i<i'} \text{sign}\Big[\Big(R^{(n)}_{1i;\pm}\mathbf{S}^{(n)}_{1i;\pm} - R^{(n)}_{1i';\pm}\mathbf{S}^{(n)}_{1i';\pm}\Big) \times \Big(R^{(n)}_{2i;\pm}\mathbf{S}^{(n)}_{2i;\pm} - R^{(n)}_{2i';\pm}\mathbf{S}^{(n)}_{2i';\pm}\Big)'\Big], \tag{3.3}$$

where $\text{sign}\big[\mathbf{M}\big]$ stands for the matrix collecting the signs of the entries of a real matrix $\mathbf{M}$, and

$$\underset{\sim}{\mathbf{W}}^{(n)}_{J} := \frac{1}{n}\sum_{i=1}^{n} J_1\Big(\frac{R^{(n)}_{1i;\pm}}{n_R+1}\Big)J_2\Big(\frac{R^{(n)}_{2i;\pm}}{n_R+1}\Big)\mathbf{S}^{(n)}_{1i;\pm}\mathbf{S}^{(n)\prime}_{2i;\pm}, \tag{3.4}$$

where the *score functions* $J_k : [0, 1) \to \mathbb{R}$, $k = 1, 2$ are the square-integrable differences of two monotone increasing functions, with

$$0 < \sigma^2_{J_k} := \int_0^1 J_k^2(u)\mathrm{d}u < \infty. \tag{3.5}$$

Those matrices defined in (3.1)–(3.4) clearly constitute matrices of cross-covariance measurements based on center-outward ranks and signs (for $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}$, signs only). For $d_1 = 1 = d_2$, it is easily seen that $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}$, $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{S}}$, and $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{K}}$, up to scaling constants, reduce to the quadrant, Spearman, and Kendall test statistics, while $\underset{\sim}{\mathbf{W}}^{(n)}_{J}$ yields a score-based extension of Spearman's correlation coefficient.

3.2 Asymptotic representation and asymptotic normality

Each of the rank-based matrices defined in (3.1)–(3.4) has an asymptotic representation in terms if i.i.d. variables. More precisely, defining $\mathbf{S}_{ki;\pm}$ as $\mathbf{F}_{k;\pm}(\mathbf{X}_{ki})/\big\|\mathbf{F}_{k;\pm}(\mathbf{X}_{ki})\big\|$ if $\mathbf{F}_{k;\pm}(\mathbf{X}_{ki}) \neq \mathbf{0}$ and $\mathbf{0}$ otherwise for $k = 1, 2$, let

$$\mathbf{W}^{(n)}_{\text{sign}} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{S}_{1i;\pm}\mathbf{S}'_{2i;\pm}, \tag{3.6}$$

$$\mathbf{W}^{(n)}_{\text{S}} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{F}_{1;\pm}(\mathbf{X}_{1i})\mathbf{F}'_{2;\pm}(\mathbf{X}_{2i}), \tag{3.7}$$

$$\mathbf{W}^{(n)}_{\text{K}} := \binom{n}{2}^{-1}\sum_{i<i'} \text{sign}\Big[\Big(\mathbf{F}_{1;\pm}(\mathbf{X}_{1i}) - \mathbf{F}_{1;\pm}(\mathbf{X}_{1i'})\Big) \times \Big(\mathbf{F}_{2;\pm}(\mathbf{X}_{2i}) - \mathbf{F}_{2;\pm}(\mathbf{X}_{2i'})\Big)'\Big], \tag{3.8}$$

and

$$\mathbf{W}^{(n)}_{J} := \frac{1}{n}\sum_{i=1}^{n} J_1\Big(\big\|\mathbf{F}_{1;\pm}(\mathbf{X}_{1i})\big\|\Big)J_2\Big(\big\|\mathbf{F}_{2;\pm}(\mathbf{X}_{2i})\big\|\Big)\mathbf{S}_{1i;\pm}\mathbf{S}'_{2i;\pm}. \tag{3.9}$$

The following asymptotic representation results then hold under the null hypothesis of independence (hence, also under contiguous alternatives).

Proposition 3.1. *Under the null hypothesis of independence, as n_R and n_S tend to infinity,* $\text{vec}\big(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}} - \mathbf{W}^{(n)}_{\text{sign}}\big)$, $\text{vec}\big(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{S}} - \mathbf{W}^{(n)}_{\text{S}}\big)$, $\text{vec}\big(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{K}} - \mathbf{W}^{(n)}_{\text{K}}\big)$, *and, provided that J_1 and J_2 are the square-integrable differences of two monotone increasing functions,* $\text{vec}\big(\underset{\sim}{\mathbf{W}}^{(n)}_{J} - \mathbf{W}^{(n)}_{J}\big)$ *are $o_{\text{q.m.}}(n^{-1/2})$.*

The asymptotic normality for $\text{vec}\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}$, $\text{vec}\underset{\sim}{\mathbf{W}}^{(n)}_{\text{S}}$, $\text{vec}\underset{\sim}{\mathbf{W}}^{(n)}_{\text{K}}$, and $\text{vec}\underset{\sim}{\mathbf{W}}^{(n)}_{J}$ follows immediately from the asymptotic representation results and the standard central-limit behavior of $\text{vec}\mathbf{W}^{(n)}_{\text{sign}}$, $\text{vec}\mathbf{W}^{(n)}_{\text{S}}$, $\text{vec}\mathbf{W}^{(n)}_{\text{K}}$, and $\text{vec}\mathbf{W}^{(n)}_{J}$.

Proposition 3.2. *Under the null (independence) hypothesis, as n_R and n_S tend to infinity, $n^{1/2}\text{vec}\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}$, $n^{1/2}\text{vec}\underset{\sim}{\mathbf{W}}^{(n)}_{\text{S}}$, $n^{1/2}\text{vec}\underset{\sim}{\mathbf{W}}^{(n)}_{\text{K}}$, and $n^{1/2}\text{vec}\underset{\sim}{\mathbf{W}}^{(n)}_{J}$ are asymptotically normal with mean vectors $\mathbf{0}_{d_1d_2}$ and covariance matrices*

$$\frac{1}{d_1d_2}\mathbf{I}_{d_1d_2}, \quad \frac{1}{9d_1d_2}\mathbf{I}_{d_1d_2}, \quad \frac{4}{9}\mathbf{I}_{d_1d_2}, \quad \textit{and} \quad \frac{\sigma^2_{J_1}\sigma^2_{J_2}}{d_1d_2}\mathbf{I}_{d_1d_2},$$

respectively.

3.3 Center-outward sign, Spearman, Kendall, and score tests

Associated with $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}$, $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{S}}$, $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{K}}$, and $\underset{\sim}{\mathbf{W}}^{(n)}_{J}$ are the sign, Spearman, Kendall, and score test statistics

$$\underset{\sim}{T}^{(n)}_{\text{sign}} := nd_1d_2\big\|\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}\big\|^2_{\text{F}}, \quad \underset{\sim}{T}^{(n)}_{\text{S}} := 9nd_1d_2\big\|\underset{\sim}{\mathbf{W}}^{(n)}_{\text{S}}\big\|^2_{\text{F}},$$

$$\underset{\sim}{T}^{(n)}_{\text{K}} := \frac{9n}{4}\big\|\underset{\sim}{\mathbf{W}}^{(n)}_{\text{K}}\big\|^2_{\text{F}}, \quad \text{and} \quad \underset{\sim}{T}^{(n)}_{J} := \frac{nd_1d_2}{\sigma^2_{J_1}\sigma^2_{J_2}}\big\|\underset{\sim}{\mathbf{W}}^{(n)}_{J}\big\|^2_{\text{F}},$$

respectively, where $\|\mathbf{M}\|_{\text{F}}$ stands for the Frobenius norm of a matrix $\mathbf{M}$, and $\sigma^2_{J_k}$, $k = 1, 2$ are defined as in (3.5).

In view of the asymptotic normality results in Proposition 3.2, the tests (denoted respectively by $\psi^{(n)}_{\text{sign}}$, $\psi^{(n)}_{\text{S}}$, $\psi^{(n)}_{\text{K}}$, and $\psi^{(n)}_{J}$) rejecting the null hypothesis of independence whenever $\underset{\sim}{T}^{(n)}_{\text{sign}}$, $\underset{\sim}{T}^{(n)}_{\text{S}}$, $\underset{\sim}{T}^{(n)}_{\text{K}}$, or $\underset{\sim}{T}^{(n)}_{J}$ exceed the $(1-\alpha)$-quantile $\chi^2_{d_1d_2;1-\alpha}$ of a chi-square distribution with d_1d_2 degrees of freedom have asymptotic level α. These tests are strictly distribution-free, however, and exact critical values can be computed or simulated as well. The tests based on $\underset{\sim}{T}^{(n)}_{\text{sign}}$, $\underset{\sim}{T}^{(n)}_{\text{S}}$, and $\underset{\sim}{T}^{(n)}_{\text{K}}$ are multivariate extensions of the traditional quadrant, Spearman, and Kendall tests, respectively, to which they reduce for $d_1 = 1 = d_2$.

4 Local asymptotic power

While there is only one way for two random vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ to be independent, their mutual dependence can take many forms. The classical benchmark, in testing for bivariate independence, is a "local" form of an independent component analysis model that goes back to Konijn (1956). A multivariate extension of such alternatives has been considered also by Gieser and Randles (1997), Taskinen, Kankainen, and Oja (2003) and Hallin and Paindaveine (2008) in the elliptical context. We extend it further here to more general, non-elliptical situations.

4.1 Generalized Konijn alternatives

Let $\mathbf{X}^* = (\mathbf{X}_1^{*\prime}, \mathbf{X}_2^{*\prime})'$, where $\mathbf{X}_1^*$ and $\mathbf{X}_2^*$ be mutually independent random vectors, with absolutely continuous distributions P_1 over $\mathbb{R}^{d_1}$ and P_2 over $\mathbb{R}^{d_2}$ and densities f_1 and f_2, respectively; then $\mathbf{X}^*$ has density $f = f_1 f_2$ over $\mathbb{R}^d$. Consider

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} := \mathbf{M}_\delta \begin{pmatrix} \mathbf{X}_1^* \\ \mathbf{X}_2^* \end{pmatrix} := \begin{pmatrix} (1-\delta)\mathbf{I}_{d_1} & \delta\mathbf{M}_1 \\ \delta\mathbf{M}_2 & (1-\delta)\mathbf{I}_{d_2} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1^* \\ \mathbf{X}_2^* \end{pmatrix} \qquad (4.1)$$

where $\delta \in \mathbb{R}$ and $\mathbf{M}_1 \in \mathbb{R}^{d_1 \times d_2}$, $\mathbf{M}_2 \in \mathbb{R}^{d_2 \times d_1}$ are nonzero. For given P_1, P_2, $\mathbf{M}_1$, and $\mathbf{M}_2$, the distribution $\mathrm{P}^{\mathbf{X}}$ of $\mathbf{X}$ belongs to a one-parameter family $\mathcal{P}^{\mathbf{X}} := \{\mathrm{P}^{\mathbf{X}}_\delta \,|\, \delta \in \mathbb{R}\}$.

On f_1 and f_2, we make the following assumption.

Assumption 4.1.

(K1) The densities f_1 and f_2 are such that

$$\int_{\mathbb{R}^{d_k}} \mathbf{x} f_k(\mathbf{x}) \mathrm{d}\mathbf{x} = \mathbf{0} \quad \textit{and}$$
$$0 < \int_{\mathbb{R}^{d_k}} \mathbf{x}\mathbf{x}' f_k(\mathbf{x}) \mathrm{d}\mathbf{x} =: \mathbf{\Sigma}_k < \infty, \quad k = 1, 2.$$

(K2) The functions $\mathbf{x}_k \mapsto (f_k(\mathbf{x}_k))^{1/2}$, $k = 1, 2$ admit quadratic mean partial derivatives[7]

$$D_\ell[(f_k)^{1/2}], \quad \ell = 1, \dots, d_k,\ k = 1, 2.$$

(K3) Letting

$$\boldsymbol{\varphi} := (\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2')' := (\varphi_{1;1}, \dots, \varphi_{1;d_1}, \varphi_{2;1}, \dots, \varphi_{2;d_1})'$$

with

$$\varphi_{k;\ell} := -2D_\ell[(f_k)^{1/2}]/(f_k)^{1/2} \overset{a.e.}{=} -\partial_\ell f_k / f_k, \quad \ell = 1, \dots, d_k,\ k = 1, 2,$$

it holds that, for $k = 1, 2$ and $\ell = 1, \dots, d_k$, we have $0 < \int_{\mathbb{R}^{d_k}} \left(\varphi_{k;\ell}(\mathbf{x})\right)^2 < \infty$, and[8]

$$\mathcal{J}_k := \mathrm{Var}\left(\mathbf{X}_k^{*\prime} \boldsymbol{\varphi}_k(\mathbf{X}_k^*)\right) = \int_{\mathbb{R}^{d_k}} \left(\mathbf{x}'\boldsymbol{\varphi}_k(\mathbf{x}) - d_k\right)^2 f_k(\mathbf{x}) \mathrm{d}\mathbf{x} < \infty.$$

It should be stressed, however, that these assumptions are not to be imposed on the observations in order for our tests to be valid but only intend to provide an analytically convenient benchmark for the comparison of local power. Let

$$\boldsymbol{\mathcal{I}}_k := \int_{\mathbb{R}^{d_k}} \boldsymbol{\varphi}(\mathbf{x}) \boldsymbol{\varphi}'(\mathbf{x}) f_k(\mathbf{x}) \mathrm{d}\mathbf{x} < \infty.$$

Under $\mathrm{P}^{\mathbf{X}}_0$, $\mathbf{X}_1 = \mathbf{X}_1^*$ and $\mathbf{X}_2 = \mathbf{X}_2^*$ are mutually independent; for $\delta \neq 0$, call $\mathrm{P}^{\mathbf{X}}_\delta$ a (generalized) *Konijn alternative* to $\mathrm{P}^{\mathbf{X}}_0$. Sequences of the form $\mathrm{P}^{\mathbf{X}}_{n^{-1/2}\tau}$ with $\tau \neq 0$, as we shall see, constitute local alternatives to the null hypothesis of independence in a sample of size n. More precisely, the following LAN property holds in the vicinity of $\delta = 0$.

Proposition 4.1. *Let P_1 and P_2 satisfy Assumption 4.1. Then, denoting by $\mathbf{X}^{(n)} := (\mathbf{X}_1, \dots, \mathbf{X}_n)$, $n \in \mathbb{N}$ a triangular array of n independent copies of $\mathbf{X} \sim \mathrm{P}^{\mathbf{X}}_0$, for given nonzero $\mathbf{M}_1$ and $\mathbf{M}_2$, the family $\mathcal{P}_{\mathbf{X}}$ of Konijn alternatives is LAN at $\delta = 0$ with root-n contiguity rate, central sequence*

$$\Delta^{(n)}(\mathbf{X}^{(n)}) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big[\mathbf{X}_{1i}' \mathbf{M}_2' \boldsymbol{\varphi}_2(\mathbf{X}_{2i}) + \mathbf{X}_{2i}' \mathbf{M}_1' \boldsymbol{\varphi}_1(\mathbf{X}_{1i}) - \Big(\mathbf{X}_{1i}' \boldsymbol{\varphi}_1(\mathbf{X}_{1i}) - d_1\Big) - \Big(\mathbf{X}_{2i}' \boldsymbol{\varphi}_2(\mathbf{X}_{2i}) - d_2\Big) \Big] \qquad (4.2)$$

and Fisher information

$$\gamma^2 := \mathcal{J}_1 + \mathcal{J}_2 + \mathrm{vec}'(\mathbf{\Sigma}_1)\,\mathrm{vec}(\mathbf{M}_2' \boldsymbol{\mathcal{I}}_2 \mathbf{M}_2) + \mathrm{vec}'(\mathbf{\Sigma}_2)\,\mathrm{vec}(\mathbf{M}_1' \boldsymbol{\mathcal{I}}_1 \mathbf{M}_1) + \mathrm{tr}(\mathbf{M}_1 \mathbf{M}_2) + \mathrm{tr}(\mathbf{M}_2 \mathbf{M}_1). \qquad (4.3)$$

Namely, under $\mathrm{P}^{\mathbf{X}}_0$,

$$\Lambda^{(n)}(\mathbf{X}^{(n)}) := \log \frac{\mathrm{d}\mathrm{P}^{\mathbf{X}}_{n^{-1/2}\tau}}{\mathrm{d}\mathrm{P}^{\mathbf{X}}_0}(\mathbf{X}^{(n)}) = \tau \Delta^{(n)}(\mathbf{X}^{(n)}) - \frac{1}{2}\tau^2\gamma^2 + o_{\mathrm{P}}(1) \qquad (4.4)$$

and $\Delta^{(n)}(\mathbf{X}^{(n)})$ is asymptotically normal, with mean zero and variance γ^2 as $n \to \infty$.

4.2 Limiting distributions and Pitman efficiencies

In this section, we aim at establishing elliptical Chernoff–Savage and Hodges–Lehmann results for our center-outward tests based on van der Waerden and Wilcoxon scores compared to Wilks' test, respectively; compare Chernoff and Savage (1958) and Hodges and Lehmann (1956). To this end, we first derive the limiting distributions of $\underset{\sim}{T}^{(n)}_J$ and $\underset{\sim}{T}^{(n)}_{\text{K}}$ under the sequence of alternatives $\mathrm{P}^{\mathbf{X}}_{n^{-1/2}\tau}$.

[7] Existence of quadratic mean partial derivatives is equivalent to quadratic mean differentiability; this was shown in Lind and Roussas (1972) and independently rediscovered by Garel and Hallin (1995, Lemma 2.1).

[8] Integration by parts yields $\int_{\mathbb{R}^{d_k}} \boldsymbol{\varphi}_k(\mathbf{x}) f_k(\mathbf{x}) \mathrm{d}\mathbf{x} = \mathbf{0}$, $\int_{\mathbb{R}^{d_k}} \mathbf{x}' \boldsymbol{\varphi}_k(\mathbf{x}) f_k(\mathbf{x}) \mathrm{d}\mathbf{x} = d_k$, and $\int_{\mathbb{R}^{d_k}} \mathbf{x} \boldsymbol{\varphi}_k(\mathbf{x})' f_k(\mathbf{x}) \mathrm{d}\mathbf{x} = \mathbf{I}_{d_k}$, for $k = 1, 2$; see also Garel and Hallin (1995, page 555).

Proposition 4.2. *Let P_1 and P_2 satisfy Assumption 4.1. Then, if observations are n independent copies with distribution $\mathrm{P}^{\mathbf{X}}_{n^{-1/2}\tau}$, for given nonzero $\mathbf{M}_1$ and $\mathbf{M}_2$,*

(i) the limiting distribution of the test statistic $\underset{\sim}{T}^{(n)}_J$ is noncentral chi-square with d_1d_2 degrees of freedom and noncentrality parameter

$$\frac{\tau^2 d_1 d_2}{\sigma^2_{J_1}\sigma^2_{J_2}}\left\|\mathrm{E}_{H_0}\left[\mathbf{J}_1(\mathbf{F}_{1;\pm}(\mathbf{X}_1))\mathbf{R}\mathbf{J}_2(\mathbf{F}_{2;\pm}(\mathbf{X}_2))'\right]\right\|^2_{\mathrm{F}},$$

where $\mathbf{R} := \mathbf{X}_1'\mathbf{M}_2'\boldsymbol{\varphi}_2(\mathbf{X}_2) + \mathbf{X}_2'\mathbf{M}_1'\boldsymbol{\varphi}_1(\mathbf{X}_1)$ and

$$\mathbf{J}_k(\mathbf{u}) := J_k(\|\mathbf{u}\|)\frac{\mathbf{u}}{\|\mathbf{u}\|}\mathbf{1}_{[\|\mathbf{u}\|\neq 0]}, \quad \mathbf{u}\in\mathbb{S}_d;$$

(ii) the limiting distribution of the Kendall test statistic $\underset{\sim}{T}^{(n)}_{\mathrm{K}}$ is noncentral chi-square with d_1d_2 degrees of freedom and noncentrality parameter

$$9\tau^2\left\|\mathrm{E}_{H_0}\left[\mathbf{F}^{\Box}_{1;\pm}(\mathbf{X}_1)\mathbf{R}\mathbf{F}^{\Box}_{2;\pm}(\mathbf{X}_2)'\right]\right\|^2_{\mathrm{F}},$$

where

$$\left(\mathbf{F}^{\Box}_{k;\pm}(\mathbf{X}_k)\right)_j := 2F_{kj}\left(\left(\mathbf{F}_{k;\pm}(\mathbf{X}_k)\right)_j\right) - 1$$

(recall F_{kj} denotes the cumulative distribution function of $\left(\mathbf{F}_{k;\pm}(\mathbf{X}_k)\right)_j$).

Suppose that all the conditions in Proposition 4.2 hold. Then the limiting alternative distribution of Wilks' (log) likelihood ratio test statistic is also noncentral chi-square, with d_1d_2 degrees of freedom and noncentrality parameter

$$\tau^2\left\|\boldsymbol{\Sigma}_1^{1/2}\mathbf{M}_2'\boldsymbol{\Sigma}_2^{-1/2} + \boldsymbol{\Sigma}_1^{-1/2}\mathbf{M}_1\boldsymbol{\Sigma}_2^{1/2}\right\|^2_{\mathrm{F}};$$

see, e.g., page 919 of Taskinen, Oja, and Randles (2005).

Now we are ready to compute the asymptotic relative efficiencies of our center-outward rank tests with respect to Wilks' likelihood ratio test.

Proposition 4.3. *Let P_1 and P_2 be elliptically symmetric distributions, namely, admit densities of the form*

$$f_k(\mathbf{x}_k) \propto (\det(\boldsymbol{\Sigma}_k))^{-1/2}\phi_k\left(\sqrt{\mathbf{x}_k'\boldsymbol{\Sigma}_k^{-1}\mathbf{x}_k}\right), \quad k=1,2,$$

satisfying Assumption 4.1. Then, the Pitman asymptotic relative efficiency (ARE) of the center-outward test based on score functions J_k, $k=1,2$ with respect to Wilks' test (denoted by $\psi^{(n)}_{\mathcal{N}}$) is

$$\mathrm{ARE}(\psi^{(n)}_J, \psi^{(n)}_{\mathcal{N}}) = \frac{\left\|D_1C_2\boldsymbol{\Sigma}_1^{1/2}\mathbf{M}_2'\boldsymbol{\Sigma}_2^{-1/2} + D_2C_1\boldsymbol{\Sigma}_1^{-1/2}\mathbf{M}_1\boldsymbol{\Sigma}_2^{1/2}\right\|^2_{\mathrm{F}}}{d_1d_2\sigma^2_{J_1}\sigma^2_{J_2}\left\|\boldsymbol{\Sigma}_1^{1/2}\mathbf{M}_2'\boldsymbol{\Sigma}_2^{-1/2} + \boldsymbol{\Sigma}_1^{-1/2}\mathbf{M}_1\boldsymbol{\Sigma}_2^{1/2}\right\|^2_{\mathrm{F}}},$$

where

$$C_k \equiv C_k(J_k,\phi_k) := \mathrm{E}[J_k^{-1}(U)\rho_k(\tilde{F}_k^{-1}(U))],$$
$$D_k \equiv D_k(J_k,\phi_k) := \mathrm{E}[J_k^{-1}(U)\tilde{F}_k^{-1}(U))],$$

$\rho_k := -\phi_k'/\phi_k$, $\tilde{F}_k$ denotes the cumulative distribution function of $\|\mathbf{Y}_k\|$ with $\mathbf{Y}_k := \boldsymbol{\Sigma}_k^{-1/2}\mathbf{X}_k$, and U stands for a random variable uniformly distributed over $(0,1)$. In particular, if $\boldsymbol{\Sigma}_1\mathbf{M}_2' = \mathbf{M}_1\boldsymbol{\Sigma}_2$, we have

(i) $\mathrm{ARE}(\psi^{(n)}_{J^{\mathrm{vdW}}}, \psi^{(n)}_{\mathcal{N}}) \geq 1$, where J_k^{vdW}, $k=1,2$ are the van der Waerden score functions $J_k^{\mathrm{vdW}}(u) := \left(F^{-1}_{\chi^2_{d_k}}(u)\right)^{1/2}$ with $F_{\chi^2_d}$ the χ^2_d cumulative distribution function;

(ii) $\mathrm{ARE}(\psi^{(n)}_{J^{\mathrm{W}}}, \psi^{(n)}_{\mathcal{N}}) \geq \Omega(d_1,d_2) \geq 9/16$, where the Wilcoxon score functions are defined as $J_k^{\mathrm{W}}(u) := u$ for $k=1,2$, and

$$\Omega(d_1,d_2) := \frac{9(2c^2_{d_1}+d_1-1)^2(2c^2_{d_2}+d_2-1)^2}{1024d_1d_2c^2_{d_1}c^2_{d_2}},$$
$$c_d := \inf\left\{x>0 \,\middle|\, \left(\sqrt{x}B_{\sqrt{2d-1}/2}(x)\right)' = 0\right\},$$
$$B_a(x) := \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+a+1)}\left(\frac{x}{2}\right)^{2m+a}.$$

Gieser (1993) notices that the Pitman ARE depends on the underlying covariance structure ($\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$) for $\mathbf{X}_1$ and $\mathbf{X}_2$ with elliptically symmetric distributions, while most the existing literature (e.g. Gieser (1993), Gieser and Randles (1997), Taskinen, Kankainen, and Oja (2003, 2004), Taskinen, Oja, and Randles (2005), Hallin and Paindaveine (2008) and Deb, Bhattacharya, and Sen (2021)) focuses on the spherically symmetric case. The proposition above fills this gap by providing the explicit formula of ARE with general $\boldsymbol{\Sigma}_k$'s. Claim (i) shows Pitman non-admissibility under ellipticity of Wilks' test, which is uniformly dominated by our center-outward test with van der Waerden scores, for elliptically symmetric distributions. This is comparable with Theorem 4.1 in Deb, Bhattacharya, and Sen (2021). Claim (ii) is a multivariate extension of Hodges and Lehmann (1956)'s result; the infimum of $\Omega(d_1,d_2)$, 9/16, is achieved when $d_1,d_2\to\infty$. One can find more numerical values of $\Omega(d_1,d_2)$ for fixed d_1,d_2 in Hallin and Paindaveine (2008, Table 3).

5 Conclusion

Optimal transport provides an entirely new approach to rank-based statistical inference in dimension $d\geq 2$. The new multivariate ranks retain many of the favorable properties one is used to with the classical univariate ranks. Here, we demonstrate how the new multivariate ranks can be used for a definition of multivariate versions of popular rank correlations such as Kendall's tau or Spearman's rho. We show how the new multivariate rank correlations yield fully distribution-free, yet powerful and computationally efficient tests of independence. A highlight of our results is the fact that the use of van der Waerden scores allows one to design a nonparametric test whose asymptotic efficiency under arbitrary elliptical densities never drops below that of Wilks' test—not even under a Gaussian model.

Acknowledgement

This project has received funding from the United States NSF Grants DMS-1712536 and SES-2019363 and European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 883818).

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